Magnetic fields and accretion discs around Kerr black holes

1983 J. Phys. A: Math. Gen. 16 2077

(http://iopscience.iop.org/0305-4470/16/9/029)

J. Phys. A: Math. Gen. 16 (1983) 2077–2086. Printed in Great Britain

Magnetic fields and accretion discs around Kerr black holes

P J Wiita†‡, C V Vishveshwara‡, M J Siah† and B R Iyer‡

† Department of Astronomy and Astrophysics, University of Pennsylvania, Philadelphia, PA 19104, USA
‡ Raman Research Institute, Bangalore 560080, India

Received 2 August 1982, in final form 12 January 1983

Abstract. We consider some aspects of accretion onto a rotating black hole immersed in a uniform magnetic field aligned with the angular momentum axis of the black hole. We specialise to motion in the equatorial plane, and calculate the 'Keplerian' angular momentum distribution and the marginally stable orbits. Using an unorthodox definition of the binding energy made necessary by an unphysical infinity induced by the assumed constancy of the magnetic field, we can calculate the marginally bound orbits and the efficiency of mass-to-energy conversion. When hydrodynamic accretion is considered the effects of the magnetic field are invariably quite small. For test particles, the magnetic field can significantly increase the efficiency, but this increase lessens as the specific angular momentum of the black hole rises.

1. Introduction

Accretion discs around supermassive black holes provide popular models for the engines of quasars and other active galactic nuclei (for a review see Wiita 1982b). Over the past few years a class of thick accretion disc models has been developed which can produce high luminosities and collimated beams of radiation and plasma (Lynden-Bell 1978, Paczyński and Wiita 1980, Jaroszyński *et al* 1980, Abramowicz and Piran 1980, Sikora and Wilson 1981, Nityananda and Narayan 1982). These models may be of great interest considering how frequently jets are being discovered in quasars and the central regions of radiogalaxies (e.g. Kellerman and Pauliny-Toth 1981).

Although the inner edges of accretion discs are normally taken to lie at the innermost stable circular orbit, r_{ms} ($=3r_g=6m$, for a Schwarzschild black hole of mass m, in units where $G=c=1$), in these thick disc models the inner edge can approach the marginally bound orbit, r_{mb} ($=2r_g$ for the Schwarzschild case) as a non-Keplerian angular momentum distribution can be set up (Abramowicz *et al* 1978). Magnetic fields will certainly be present in the material being accreted by the black hole (Bisnovatyi-Kogan 1979) but until the recent work of Dadhich and Wiita (1982, to be referred to as DW) on the Schwarzschild case little attention had been paid to the questions of how magnetic fields affect the values of r_{mb} and r_{ms}. They used the Ernst (1976) metric to model a black hole in a uniform magnetic field and the effective potential derived by Dadhich *et al* (1979) to show that if the flow is basically hydrodynamical, the efficiency of mass-to-energy conversion cannot be significantly changed, as r_{ms} is not altered very much. In this case r_{mb} is not reduced very much either, so that the opening angle of the funnels cannot be made significantly smaller and the collimation of ejected material is not greatly improved. On the other hand, DW showed that test particles (with a large charge/mass ratio) can have their rest masses converted to energy with an efficiency approaching unity, as both r_{ms} and r_{mb} become arbitrarily close to r_g in this static geometry.

In this paper we shall consider the more general case of a rotating black hole immersed in a uniform magnetic field of strength B, and we shall assume that $|Bm| \ll 1$ so that the mass-energy of the field is small compared with that of the Kerr black hole. Prasanna and Vishveshwara (1978, to be referred to as PV) analysed trajectories of charged particles in equatorial orbits around Kerr black holes surrounded by both uniform and dipole magnetic fields. We will use the PV effective potential to solve numerically for the 'Keplerian' angular momentum distribution, r_{ms}, r_{mb} and the efficiency at r_{ms}. The complexity of the results precluded the generation of analytic approximations similar to those found in DW. We refer the reader to Prasanna (1980) and DW for a discussion of the few papers which touch on related problems.

2. Keplerian angular momentum distribution

The Kerr metric in Boyer–Lindquist coordinates can be written as

$$ds^2 = -(1-2mr\Sigma^{-1})\,dt^2 - 4mra\Sigma^{-1}\sin^2\theta\,dt\,d\varphi + \Sigma\Delta^{-1}\,dr^2 + \Sigma\,d\theta^2 + F\Sigma^{-1}\sin^2\theta\,d\varphi^2 \tag{1}$$

where a is the angular momentum parameter, $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2mr$ and $F = (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$. Wald (1974) has found the vector potential for the electromagnetic field for a stationary, axisymmetric black hole placed in an originally uniform magnetic field of strength B aligned along the black hole's symmetry axis. We follow PV in using this result in the case of a pure Kerr (no electrostatic charge) black hole:

$$\begin{aligned} A_t &= -aB[1 - mr\Sigma^{-1}(2-\sin^2\theta)] \\ A_\varphi &= \tfrac{1}{2}B\sin^2\theta\,\Sigma^{-1}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta - 4ma^2r]. \end{aligned} \tag{2}$$

Since both the gravitational and electromagnetic fields are axisymmetric and stationary there exist two Killing vectors, such that for the motion of a particle of charge e and rest mass μ, there are two constants of motion. These are the canonical angular momentum and the energy, which are respectively given by

$$U_\varphi + eA_\varphi = L, \qquad U_t + eA_t = -E, \tag{3}$$

where all quantities are normalised by division by μ, and U_φ and U_t are connected to components of the particle's proper velocity U^i.

After introducing dimensionless quantities

$$\begin{aligned} &\rho = r/m, \qquad \sigma = s/m, \qquad l = L/m, \\ &\alpha = a/m, \qquad \tau = t/m, \qquad \bar{A}_\varphi = A_\varphi/m, \end{aligned} \tag{4}$$

the radial velocity is found to be (PV, equation (23))

$$\begin{aligned} (U^\rho)^2 \equiv (d\rho/d\sigma)^2 = \rho^{-3}\{&[\rho(\rho^2+\alpha^2)+2\alpha^2](E+A_\tau)^2 \\ &- 4\alpha(E+A_\tau)(l-\bar{A}_\varphi) - (\rho-2)(l-\bar{A}_\varphi)^2 - \rho\Delta\} \end{aligned} \tag{5}$$

where we have restricted ourselves to considering motion confined to the equatorial plane, so that $\theta = \pi/2$, $\mathrm{d}\theta/\mathrm{d}\sigma = 0$. This restriction is reasonable in that the minimum values for r_{mb} and r_{ms}., and thus the determinants of the efficiency of a disc, are in the equatorial plane. In terms of the parameter

$$\lambda = eBm \tag{6}$$

the potentials of (2) that appear in (5) take the form

$$A_\tau = -\lambda\alpha(1-\rho^{-1}), \qquad \bar{A}_\varphi = \tfrac{1}{2}\lambda[\rho^2+\alpha^2(1-2\rho^{-1})]. \tag{7}$$

The effective potential for radial motion is found by solving for the turning points of the orbits. Setting $U^\rho = 0$ in (5) yields the effective potential

$$V = \lambda\alpha(1-\rho^{-1}) + K/R \tag{8}$$

where

$$\begin{aligned} K &= \{2\alpha(l-\bar{A}_\varphi) + \Delta^{1/2}[\rho^2(l-\bar{A}_\varphi)^2 + \rho R]^{1/2}\}, \\ R &= (\rho^3+\alpha^2\rho+2\alpha^2), \qquad \Delta = \rho^2 - 2\rho + \alpha^2. \end{aligned} \tag{9}$$

The locations of the maxima and minima for this effective potential, as well as possible test particle orbits for fixed λ, α and l, are discussed in PV.

We now seek the angular momentum distribution characterised by a balance between rotational, gravitational and electromagnetic forces (but neglecting radiation losses) that is the analogy to circular Keplerian orbits in the Newtonian case. This function, $l_K(\rho)$, is found by setting $\mathrm{d}V/\mathrm{d}\rho = 0$. After a significant amount of algebra we find that

$$\mathrm{d}V/\mathrm{d}\rho \equiv V' = T_1 + T_2(l^2 - Xl + Y)^{1/2} + T_3 l + T_4(l^2 - Wl + Z)(l^2 - Xl + Y)^{-1/2} \tag{10}$$

where the coefficients of l are functions solely of ρ, with α and λ taken as parameters. Explicit forms for these coefficients are

$$\begin{aligned} &X = \lambda\rho^{-1}(R-4\alpha^2), && Y = \tfrac{1}{4}\rho^{-2}[\lambda^2(R-4\alpha^2)^2 + 4\rho R], \\ &W = \lambda\rho^{-1}[2R - \alpha^2(\rho+5)], && Z = \tfrac{1}{4}\rho^{-1}\{\lambda^2(R-4\alpha^2)S + 4[2R-\alpha^2(\rho+3)]\}, \\ &S = 3\rho^2+\alpha^2, && T_1 = 2\alpha\lambda(\rho R)^{-2}[R(R-2\alpha^2) - 2\alpha^2\rho S] \\ &T_2 = \rho R^{-2}\Delta^{-1/2}[R(\rho-1) - \Delta S], && T_3 = -2\alpha S R^{-2}, \qquad T_4 = \Delta^{1/2}R^{-1}. \end{aligned} \tag{11}$$

Setting $V' = 0$, we convert (10) into an explicit quartic equation for l:

$$\begin{aligned} l^4&[(T_2+T_4)^2 - T_3^2] \\ &+ l^3[T_3^2X - 2T_1T_3 - 2(T_2+T_4)(T_2X+T_4W)] \\ &+ l^2[(T_2X+T_4W)^2 + 2(T_2+T_4)(T_2Y+T_4Z) - (T_3^2Y + T_1^2 - 2T_1T_3X)] \\ &+ l[T_1^2X - 2T_1T_3Y - 2(T_2X+T_4W)(T_2Y+T_4Z)] \\ &+ [(T_2Y+T_4Z)^2 - T_1^2Y] = 0. \end{aligned} \tag{12}$$

Equation (12) must be solved numerically, and the real positive root corresponding to each value of $\rho > \rho_+$ (the event horizon, $\rho_+ = 1 + (1-\alpha^2)^{1/2}$) is chosen as $l_K(\rho;\alpha,\lambda)$. As we are able to express the auxiliary function K in terms of l,

$$K = \alpha\rho^{-1}[2\rho l - \lambda(R-4\alpha^2)] + \rho\Delta^{1/2}(l^2 - Xl + Y)^{1/2}$$

we can then substitute back into (8) to solve for $V(\rho; l_K(\rho))$. Because we are interested in cases with small r_{mb} and r_{ms} and high efficiency, we only consider co-rotating particles or discs; i.e. $l\alpha \geqslant 0$ (Bardeen 1970).

The binding energy (per unit mass) is typically defined as

$$b = \lim_{\rho \to \infty} V - V \equiv V_\infty - V.$$

However, in our case we find that for large ρ, $V(\rho) \cong \alpha\lambda + (1 - l\lambda + \frac{1}{4}\lambda^2\rho^2)^{1/2}$ so that $V_\infty = \infty$, and this definition is indeterminate. It is obvious on physical grounds that V_∞ diverges as we have assumed a constant magnetic field filling all space, so that even in the absence of a central mass one garners an indeterminately large energy at infinity. However, this divergence is really an artifact of that unphysical assumption, in that any realistic magnetic field must decrease at large radii (DW) rapidly enough to remove any singularity. Therefore we shall henceforth neglect this effectively arbitrary contribution to the potential and choose the definition of binding energy that holds in the absence of a magnetic field, $\mu - E$, or in our units, the binding energy per unit mass is

$$b = 1 - V. \tag{13}$$

(Note that the same argument must be made with respect to the work of DW if it is to retain its validity.) With this definition ρ_{mb} is found as the first radius for which $V = 1$ as V decreases while ρ increases above ρ_+. The marginally stable orbit is the one where $dl_K/dr = 0$ (its location is independent of the choice of definition of b). Both of these points are found numerically by following the approach of DW.

3. Plausible parameter ranges

Before discussing the values obtained for the efficiency, last stable orbit, and last bound orbit as functions of the parameters α and λ, we must discuss the likely ranges for these parameters. As far as the angular momentum of the black hole is concerned, we are naturally interested in all values from $\alpha = 0$ (Schwarzschild case) through $\alpha = 1$ (extreme Kerr case). Although numerical problems preclude solving equation (12) for both of those limiting cases, we were able to obtain useful results for $10^{-4} \leqslant \alpha \leqslant 0.999\,999\,99$; as discussed below, the agreement of these near limiting cases with the actual extremes is excellent.

The parameter λ couples the strength of the magnetic field, Bm, with the charge (per mass) of the accreting material, e. Let us first consider the magnetic field. The conversion between the dimensionless variable Bm and physical units for the magnetic field is

$$B_G = (c^4/G^{3/2}M)Bm = 2{\cdot}36 \times 10^{19}(M_\odot/M)(Bm)\,\mathrm{G}, \tag{14}$$

where M is the mass of the black hole in grams. A plausible constraint on Bm comes from the demand that the pressure due to the magnetic field does not exceed the sum of the gas and radiation pressures in a fluid disc. As shown in DW, the models of Wiita (1982a) can be used to find reasonable upper limits to the total pressure, and therefore, the magnetic field. The limit they obtain is $|Bm| < 10^{-4}$. Unfortunately, Wiita's models assume a Schwarzschild black hole and no equivalent calculations have been performed for Kerr black holes. However, the evidence available (Jaroszyński *et al* 1980, Sikora 1981) indicates that the physical parameters in a disc around a rotating black hole will not differ very much from those around a non-rotating one, so we feel that the above limit is a reasonable one, especially in view of the fact that it was based on the most extreme models. In most circumstances we expect an even weaker field to exist in the vicinity of the hole ($Bm \leqslant 10^{-6}$); for a supermassive black hole of $10^8 M_\odot$ this would correspond to $\sim 10^5$ G.

The value of the charge to be expected is quite uncertain. As discussed in DW, we anticipate that a plasma disc will be subject to forces that produce some charge separation and thus an effective charge in the innermost region. However, expressed as a charge per unit mass in geometrical units (our e) this value should be relatively small, probably of O(1) or less. In any event, the product of charge with magnetic field (λ) should be significantly less than unity for fluid accretion. On the other hand, in the test particle approximation, very high values of e are possible (for an isolated proton, $e = 1.112 \times 10^{18}$) and we must allow for the possibility that λ is much greater than one, the situation considered by PV.

4. Numerical results

Before exploring uncharted regions of the (α, λ) parameter space, we compared our results with those of previous calculations. As mentioned above, numerical difficulties in the root finding algorithm prevented us from performing the calculations for $\alpha = 0$ or $\alpha = 1$. However, results were obtained for values of α as low as 10^{-4}, and these could then be compared with the Schwarzschild results of DW, although a difference in definition means that our λ corresponds to one-half of their product eBm. The differences in the locations of r_{mb}, r_{ms} and the value of $b(r_{ms})$ were found to differ by $\leqslant 0.2\%$ from those of DW for the equivalent λ value. Part of this discrepancy is due to the non-zero α we employed and part is due to our use of a perturbation around the Kerr metric while they used the Ernst metric.

Comparisons were also made with Thorne's (1974) results for accretion onto Kerr black holes without electromagnetic fields. When we set $\lambda = 0$ in our computations we find exact agreement with Thorne's results for the values of r_{ms} and $b(r_{ms})$ for essentially all values of α tested; he does not give values for r_{mb}. Our program could not reproduce the efficiency of 0.423 given by Thorne for $\alpha = 1$, but we find an efficiency of 0.421 for $\alpha = 0.999\,999\,99$, an agreement we consider satisfactory. Values of $\alpha = 0.9978$ and 0.9982 correspond to the 'canonical' spin-up limits for a black hole where isotropic emission or an electron-scattering atmosphere are respectively assumed (Thorne 1974), and yield efficiencies of ~ 0.32. While this limit of $\alpha = 0.998$ is valid for thin discs, Abramowicz and Lasota (1980) have shown that for a thick disc with an inner boundary approaching r_{mb} this value can be exceeded, although $\alpha = 1$ is forbidden by the third law of black hole thermodynamics. However, we expect that the decrease in efficiency caused by this shrinkage of the inner edge will more than offset the increase due to the allowed rise in angular momentum, although detailed calculations would have to be performed to verify this conjecture.

Our numerical results are summarised in tables 1–5 and figures 1–3. The tables present results for r_{mb}, r_{ms}, $l_K(r_{ms}) \equiv l_{ms}$ and b_{ms} for five values of α and various values of λ. For fixed values of $\alpha < 0.9999$ we note that r_{mb} and r_{ms} decrease smoothly as λ increases, while l_{ms} suffers a small decrease before again rising. The minimum in l_{ms} is usually achieved for $\lambda \sim 10^{-1}$.

Table 1. Accretion parameters as functions of λ for $\alpha = 0.1$.

λ	r_{mb}	r_{ms}	l_{ms}/μ	$b_{ms}/\mu c^2$
0	3.797 366 30	5.669 302 48	3.367 110	0.060 634 45
10^{-6}	3.797 353 79	5.669 302 48	3.367 105	0.060 635 94
10^{-4}	3.796 070 49	5.669 299 10	3.366 575	0.060 783 68
10^{-2}	3.683 038 56	5.635 586 93	3.320 367	0.074 720 22
10^{-1}	3.194 622 73	4.536 380 53	3.260 053	0.154 147 90
0.2	2.968 382 95	3.868 558 00	3.405 428	0.203 565 42
1.0	2.478 607 88	2.687 908 89	5.601 304	0.352 916 49
2.0	2.303 275 41	2.399 384 30	7.008 664	0.403 783 25
3.0	2.220 616 47	2.281 698 57	8.099 709	0.413 383 58
5.0	2.140 965 06	2.175 753 35	12.953 676	0.385 369 49
10.0	2.070 985 11	2.087 477 81	22.822 266	0.222 533 53
15.0	2.046 826 65	2.056 201 06	32. 693 667	0.016 572 03

Table 2. Accretion parameters as functions of λ for $\alpha = 0.5$.

λ	r_{mb}	r_{ms}	l_{ms}/μ	$b_{ms}/\mu c^2$
0	2.914 2135	4.233 002 6	2.902 866	0.082 117 99
10^{-6}	2.914 2091	4.233 002 6	2.902 863	0.082 118 90
10^{-3}	2.909 9964	4.232 908 3	2.900 010	0.083 024 66
0.25	2.473 4462	3.141 205 25	2.935 259	0.171 815 51
0.4	2.372 8849	2.844 422 0	3.131 335	0.181 192 51
0.455	2.345 3033	2.769 208 25	3.209 209	0.181 747 18
0.55	2.305 1457	2.664 447 2	3.347 754	0.180 238 71
1.0	2.186 2783	2.384 784 68	4.035 618	0.147 650 96
1.1	2.169 1554	2.347 454 41	4.191 824	0.136 851 09
2.0	2.110 4549	2.156 214 48	5.617 347	0.011 196 34
2.5	2.078 5249	2.102 202 18	6.417 821	−0.071 657 99

Table 3. Accretion parameters as functions of λ for $\alpha = 0.9978$.

λ	r_{mb}	r_{ms}	l_{ms}/μ	$b_{ms}/\mu c^2$
0	1.096 0084	1.245 923 00	1.399 685	0.318 148 50
10^{-6}	1.096 0084	1.245 923 00	1.399 685	0.318 148 51
10^{-4}	1.096 0081	1.245 923 00	1.399 683	0.318 149 83
10^{-2}	1.095 9820	1.245 914 75	1.399 420	0.318 277 93
10^{-1}	1.095 7485	1.245 105 14	1.397 732	0.319 133 54
0.2	1.095 4971	1.242 748 38	1.397 280	0.319 452 92
0.3	1.095 2541	1.239 122 19	1.398 217	0.319 155 68
0.5	1.094 7915	1.229 478 42	1.403 666	0.316 967 18
1.0	1.093 7609	1.203 109 05	1.431 530	0.305 053 54
2.0	1.092 1629	1.166 063 91	1.514 575	0.268 416 26
3.5	1.090 7521	1.137 072 33	1.659 977	0.202 986 04
7.0	1.093 9376	1.108 684 34	2.020 329	0.038 315 78

Perhaps the most interesting result concerns the binding energy at r_{ms}, which corresponds to the efficiency of mass to energy conversion (neglecting swallowing of radiation by the black hole) for a thin accretion disc or for test particles. For fixed

Table 4. Accretion parameters as functions of λ for $\alpha = 0.9999$.

λ	r_{mb}	r_{ms}	l_{ms}/μ	$b_{ms}/\mu c^2$
0	1.021 4154	1.078 526 45	1.240 584	0.382 041 53
10^{-4}	1.021 4154	1.078 526 45	1.240 584	0.382 041 68
2.0	1.019 9058	1.063 655 19	1.258 645	0.373 424 05
7.0	1.019 5265	1.041 103 11	1.362 394	0.323 128 50
10.0	1.019 3608	1.035 649 74	1.429 979	0.290 169 50
15.0	1.019 1783	1.030 417 23	1.543 612	0.234 607 39
20.0	1.019 1168	1.027 315 32	1.657 309	0.178 906 99
25.0	1.019 2087	1.025 233 86	1.770 840	0.123 221 20
35.0	1.020 8001	1.022 583 03	1.997 555	0.118 931 22
40.0	1.021 1727	1.021 579 68	2.110 877	−0.043 795 86

Table 5. Accretion parameters as functions of λ for $\alpha = 0.999\,999\,99$.

λ	r_{mb}	r_{ms}	l_{ms}/μ	$b_{ms}/\mu c^2$
0	>1.000 2†	1.003 432 21	1.158 651	0.420 679 44
10^{-4}	>1.000 2†	1.003 430 21	1.158 651	0.420 679 45
0.2	>1.000 2†	1.003 429 68	1.158 651	0.420 679 72
0.9	>1.000 2†	1.003 419 56	1.158 658	0.420 675 90
50.0	>1.000 2†	1.001 455 25	1.169 415	0.415 303 32
500.0	1.000 1958	1.000 489 17	1.282 368	0.358 850 43
1000.0	1.000 1930	1.000 360 37	1.402 340	0.298 888 01
2000.0	1.000 1907	1.000 273 38	1.635 266	0.182 442 05
5000.0	1.000 1673	1.000 203 05	2.319 387	−0.159 558 40

† Numerical difficulties prevented accurate determination of these values.

values of $\alpha > 0$ we find that the efficiency rises with λ, but only until $\lambda \sim 1$; as the combined strength of the field and charge continues to increase b_{ms} falls off to values below that for $\lambda = 0$. This behaviour is illustrated in figure 1 where we plot b_{ms} against $\log \lambda$ for the five cases shown in the tables as well as the result for $\alpha = 0$ taken from DW. Both the absolute and relative increases in b_{ms} drop as α rises, for if we define

$$\delta b(\alpha) = b_{max}(\alpha, \lambda)/b(\alpha, 0) \tag{15}$$

we find that $\delta b(0) \simeq 17.485$, $\delta b(0.1) \simeq 6.818$, $\delta b(0.5) \simeq 2.213$, $\delta b(0.9978) \simeq 1.004$ and $\delta b(0.999\,999\,99) = 1.000\,000\,7$. As a matter of fact, for sufficiently large λ, no Keplerian orbits can be found for a given α, as the magnetic fields yield negative binding energies at all radii. Similar results were found by PV in their analysis of test particle orbits with fixed λ, l and α. Physically, this situation corresponds to powerful electromagnetic potentials preventing any stable orbits from existing. As this cut-off value of λ is always greater than 1 and rises both for $\alpha \to 0$ and $\alpha \to 1$, we see that this does not severely affect the plasma accretion case ($\lambda < 1$), but does limit the test particle case.

Figure 2 shows the effective potential plotted against radius for several values of α and λ. For $\lambda \ll 1$, V drops below 1 and then remains less than 1 until large ρ, implying that Keplerian orbits are viable over a wide range in radius. As λ increases, the radial range for which $V < 1$ drops significantly, implying that the outer edge of

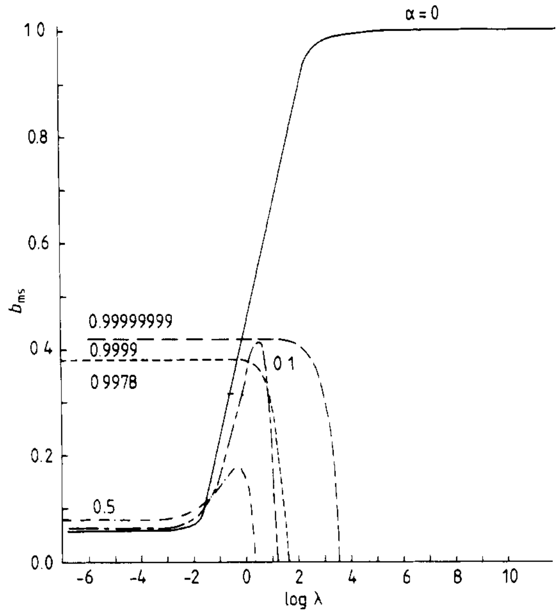

Figure 1. Efficiency of mass to energy conversion at the last stable orbit (b_{ms}) plotted against the logarithm of the product of magnetic field strength and charge (λ) for different values of the angular momentum parameter (α).

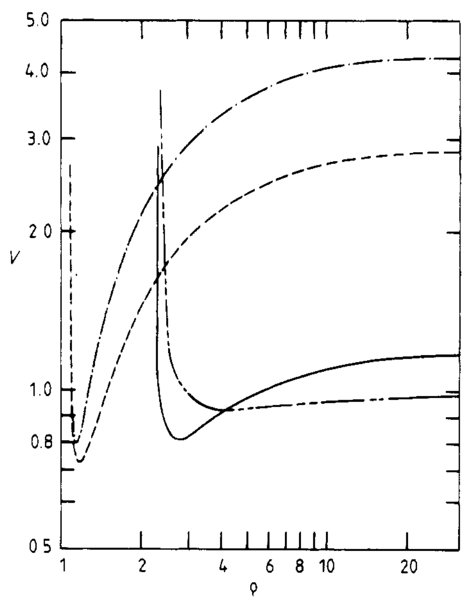

Figure 2. Effective potential, V, against dimensionless radius, ρ, plotted on a log–log scale for four sets of values of α and λ. The full curve corresponds to $\alpha = 0.5$, $\lambda = 0.445$; the broken curve to $\alpha = 0.9978$, $\lambda = 2.0$; the chain curve to $\alpha = 0.998$, $\lambda = 3.5$; and double chain curve to $\alpha = 0.5$, $\lambda = 10^{-6}$. Note that a disc only exists for $V < 1$ so that for large λ stable orbits cover a very limited range in radius.

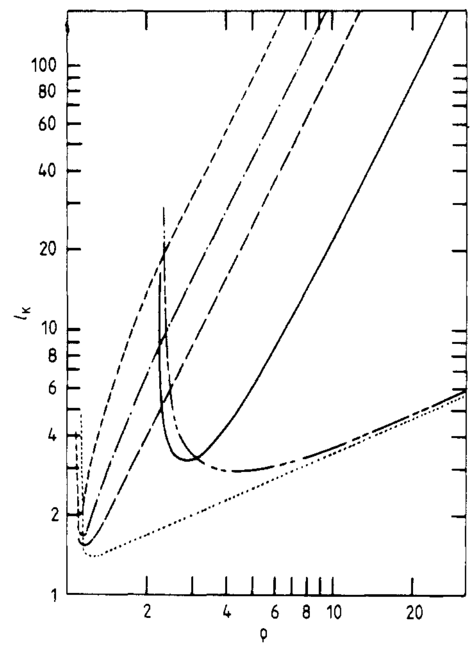

Figure 3. The specific Keplerian angular momentum, l_K, against ρ for six sets of values of α and λ, including the four cases shown in figure 2, where the types of curves corresponding to the different parameters are defined. The additional curves are: dotted, $\alpha = 0.9978$, $\lambda = 10^{-6}$; and short broken, $\alpha = 0.9978$, $\lambda = 7$.

the disc must become very close to the inner edge. This range becomes extremely narrow (as is evident for $\alpha = 0.9978$ and $\lambda = 3.5$) and eventually vanishes when no Keplerian orbits can exist. Figure 3 shows how the Keplerian angular momentum reaches a minimum at r_{ms} and then rises with radius for the cases shown in figure 2 (plus two others).

5. Conclusions

We have shown that in the Kerr geometry magnetic fields are unlikely to have significant influence upon the efficiency of astrophysically interesting accretion processes, where $\lambda \ll 1$. This result for accretion discs is basically a broadening of the conclusion reached by DW for the Ernst metric. But unique results arise in the test particle situation, $\lambda \gtrsim 1$. When the black hole is rotating, the efficiency does not continue to rise with λ beyond a certain point; rather a maximum efficiency, corresponding to a rather narrow ring of material, is reached. In the plasma accretion case, the decrease in r_{mb} is never sufficient to imply significantly narrower funnels.

Although we believe these negative results are quite strong, we must reiterate some of the simplifying assumptions that have been made in our calculations. We have specialised to the equatorial plane for a magnetic field aligned with the black hole's rotation axis; other configurations could conceivably alter the picture dramatically. However, they would be much more difficult to calculate, and we do not rate this as much of a problem. Our assumption of a uniform magnetic field is certainly too simple, and the actual field will be more complex; probably the field will be most intense in the innermost regions of the disc as it is amplified by shear in the accreting material (Bisnovatyi-Kogan 1979). But as the limits were derived on the maximum field strength, this should not change our conclusions about the values of r_{mb}, r_{ms} or b_{ms} very much. If we allowed for a decrease in field strength with radius (e.g. a dipole type field) then we expect that V would rise more slowly with distance and decent orbits could exist over larger intervals; equivalently, the cut-off value of 'λ' could rise. With a sufficiently rapidly decreasing magnetic field we expect V_∞ to be finite and we could return to the standard definition of the binding energy as $b = V_\infty - V$. This would certainly be more physically exact, and we plan to pursue this route in future work. The fact that the inner parts of thick discs will have non-Keplerian angular momentum distributions will not change the maximum efficiency (DW). the neglect of the radiation emitted by the infalling material, particularly in the test particle case, is probably the biggest source of error in our treatment, and is the most likely way in which the limits we have found could be exceeded. But again, for hydrodynamic accretion discs we anticipate this correction to be small. Although radically different pictures allow magnetic fields to play a dominant role (e.g. Blandford 1979 and references therein), we have shown that many straightforward effects of magnetism on accretion are quite small.

Acknowledgments

We thank Drs M Abramowicz and J Miller for useful criticism of a previous version of this work. PJW acknowledges the hospitality of the Raman Research Institute and the Tata Institute of Fundamental research and thanks the Smithsonian Institution for a foreign currency research travel grant.

References

Abramowicz M A, Jaroszyński M and Sikora M 1978 *Astron. Astrophys.* **63** 221–4
Abramowicz M A and Lasota P 1980 *Acta Astron.* **30** 35–9
Abramowicz M A and Piran T 1980 *Astrophys. J. Lett.* **241** L7–11
Bardeen J M 1970 *Nature* **226** 64–5
Bisnovatyi-Kogan G S 1979 *Riv. Nuovo Cimento* **2** No 1 pp 1–37
Blandford R D 1979 in *Active Galactic Nuclei* ed C Hazard and S Mitton (Cambridge: CUP) pp 241–54
Dadhich N, Hoenselaers C and Vishveshwara C V 1979 *J. Phys. A: Math. Gen.* **12** 215–21
Dadhich N and Wiita P J 1982 *J. Phys. A: Math. Gen.* **15** 2645–53
Ernst F J 1976 *J. Math. Phys.* **17** 54–6
Jaroszyński M, Abramowicz M A and Paczyński B 1980 *Acta Astron.* **30** 1–34
Kellerman K I and Pauliny-Toth I I K 1981 *Ann. Rev. Astron. Astrophys.* **19** 373–410
Lynden-Bell D 1978 *Phys. Scr.* **17** 185–91
Nityananda R and Narayan R 1982 *Mon. Not. R. Astron. Soc.* **201** 697–705
Paczyński B and Wiita P J 1980 *Astron. Astrophys.* **88** 23–31
Prasanna A R 1980 *Riv. Nuovo Cimento* **3** No 11 pp 1–53
Prasanna A R and Vishveshwara C V 1978 *Pramāna* **11** 359–77
Sikora M 1981 *Mon. Not. R. Astron. Soc.* **196** 257–68
Sikora M and Wilson D B 1981 *Mon. Not. R. Astron. Soc.* **197** 529–41
Thorne K S 1974 *Astrophys. J.* **191** 507–19
Wald R M 1974 *Phys. Rev.* D **10** 1680–93
Wiita P J 1982a *Astrophys. J.* **256** 666–80
——1982b *Comments Astrophys.* **9** 251–60